# The Fermi edge singularity and boundary condition changing operators

Ian Affleck† and Andreas W W Ludwig‡

† Canadian Institute for Advanced Research and Physics Department, University of British Columbia, Vancouver, BC, V6T1Z1 Canada
‡ Joseph Henry Laboratory, Princeton University, Princeton, NJ 08544, USA

**Abstract.** The boundary conformal field theory approach to quantum impurity problems is used to study the Fermi edge singularity, occurring in the x-ray adsorption probablility. The deep-hole creation operator, in the effective low-energy theory, changes the boundary condition on the conduction electrons. By a conformal mapping, the dimension of such an operator is related to the ground-state energy for a finite system with *different* boundary conditions at the two ends. The Fermi edge singularity is solved, using this method, for the Luttinger liquid including backscattering and for the multi-channel Kondo problem.

## 1. Introduction

We have recently developed a new method for studying quantum impurity problems based on conformal field theory with boundaries [1]. Using this method we have re-derived known results and, in some cases obtained new results, on the multi-channel [2] and two-impurity Kondo problem [3], impurities in spin chains [4], magnetic-monopole–baryon systems [5] and tunnelling in quantum wires [6]. The purpose of the present paper is to discuss the extension of our method to another type of quantum impurity problem which is exemplified by the Fermi edge singularity [7]. The standard simplified Hamiltonian for this problem is

$$H = \sum_k \epsilon_k a_k^\dagger a_k + E_0 b^\dagger b + \sum_{k,k'} V_{k,k'} a_k^\dagger a_{k'} b b^\dagger. \tag{1.1}$$

Here $a_k$ annihilates a conduction-band electron and $b$ annihilates the ionic deep core electron. One is interested in calculating the two-point Green function for the deep core operator $b$ and the operator $b a_k^\dagger$ which creates a core hole and a conduction electron; the latter transition is affected by x-ray absorption. Since the Hamiltonian of equation (1.1) commutes with the core electron number $b^\dagger b$, the Hilbert space separates into two sectors with the hole absent or present. When there is no hole, we simply obtain the free conduction-electron Hamiltonian

$$H_0 = \sum_k \epsilon_k a_k^\dagger a_k. \tag{1.2}$$

When the hole is present, the conduction electrons also feel a scattering potential, $V$:

$$H_1 = \sum_k \epsilon_k a_k^\dagger a_k + \sum_{k,k'} V_{k,k'} a_k^\dagger a_{k'}. \tag{1.3}$$

To calculate the deep core Green function $\langle b^{\dagger}(t)b(0)\rangle$, we must solve the time-dependent response of the conduction electrons for turning this potential on suddenly at time 0 and then turning it off again at time $t$. As $t \to \infty$ this Green function exhibits non-trivial power-law decay with the exponent depending on the potential $V$.

The singularity only depends on the behaviour of the potential at the Fermi surface. Therefore, it is entirely determined by the phase shift $\delta$ at the Fermi surface $k_F$. (In general, it depends on the phase shifts in all angular momentum channels at $k_F$. For simplicity, we focus on the case where there is only a single channel with a non-zero phase shift corresponding to a $\delta$-function potential.) Since the dependence of the phase shift on $k - k_F$ is irrelevant, the effective low-energy Hamiltonian for the problem has a constant phase shift for $-\Lambda < k - k_F < \Lambda$, where $\Lambda$ is the cut-off (which obeys $\Lambda \ll k_F$). A constant phase shift corresponds to a simple *boundary condition* relating incoming and outgoing waves at the impurity location

$$\psi_{\text{out}}(0) = e^{2i\delta}\psi_{\text{in}}(0). \tag{1.4}$$

Thus, the effect of acting with the core hole operator $b$ is to change the boundary conditions in the low-energy theory. The basic Fermi edge singularity problem is to calculate the scaling dimension of a boundary condition changing operator. As such, we see that it has numerous generalizations to other quantum impurity problems.

The notion of boundary condition changing operators also plays a fundamental role in Cardy's boundary conformal field theory [8]. He developed a theory of conformally invariant boundary conditions and boundary operators. Each pair of such boundary conditions corresponds to a boundary operator. However, the reverse is not true. There are also boundary operators which *do not* change the boundary conditions. We have discussed these extensively in our previous papers on quantum impurity problems [1–6,9]. The purpose of this paper is to study boundary condition changing operators in quantum impurity problems using Cardy's boundary conformal field theory.
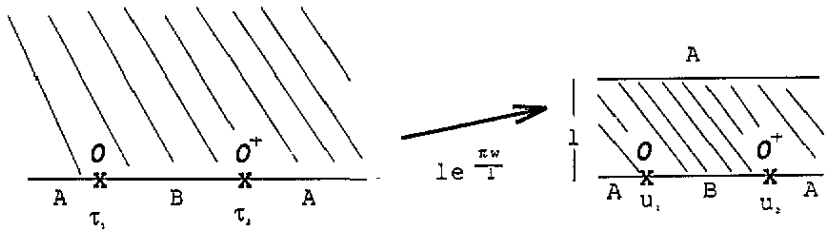


**Figure 1.** Boundary condition changing operators act at times $\tau_1$ and $\tau_2$.

The general situation is illustrated in figure 1. By s-wave projection, or its generalizations, we may formulate our quantum impurity problem in one space and one time dimension on the half-plane $r \geqslant 0$. Let us assume that, in the distant past and future, some conformally invariant boundary conditions $A$ apply. At (imaginary) time $\tau_1$, a boundary operator $\mathcal{O}$ acts which changes the boundary conditions to $B$. At time $\tau_2$, $\mathcal{O}^{\dagger}$ acts and reverts the boundary conditions to $A$ again. It is convenient to set the velocity to unity and use the complex coordinate $z = \tau + ir$. We assume that the bulk Hamiltonian is invariant under conformal transformations $z \to f(z)$ and that the boundary conditions

respect the subgroup of conformal transformations that map the real axis $z = \tau$ into itself. It is very useful to apply the conformal transformation

$$z = l e^{\pi w / l} \tag{1.5}$$

where $l$ is an arbitrary length scale. This maps the half-plane into the infinite strip $w = u + iv$, $0 \leqslant v \leqslant l$, as shown in figure 1. Note that the positive real axis $\tau > 0$ maps onto the lower boundary of the strip $v = 0$. For convenience, we have chosen the points at which the boundary operators act to obey $\tau_i > 0$ so that they both map onto the lower boundary of the strip. Hence, the boundary conditions on the top of the strip are always $A$ but on the bottom they are $B$ for $u_1 < u < u_2$ and $A$ otherwise. It appears that, in all physical cases, the ground state with the same boundary condition on the top and bottom of the strip is the absolute ground state, i.e. the primary state in the conformal tower corresponding to the identity operator.

We now wish to relate the scaling dimension of the boundary condition changing operator $O$ to the ground-state energy of the finite system with boundary condition $A$ on one side and $B$ on the other. Letting this dimension be $x$ and assuming a convenient normalization for the operator, Green's function on the half plane with boundary condition $A$ is

$$\langle A | O(\tau_1) O^\dagger(\tau_2) | A \rangle = \frac{1}{(\tau_1 - \tau_2)^{2x}}. \tag{1.6}$$

Green's function on the strip is obtained by the conformal mapping. Assuming $O$ to be primary we obtain

$$\langle AA | O(u_1) O^\dagger(u_2) | AA \rangle = \left[ \frac{(\partial z / \partial w(u_1))(\partial z / \partial w(u_2))}{[z(u_1) - z(u_2)]^2} \right]^x$$
$$= \frac{1}{[(\frac{2l}{\pi}) \sinh(\frac{\pi}{2l})(u_1 - u_2)]^{2x}}. \tag{1.7}$$

Now let $u_2 - u_1 \gg l$, giving

$$\langle AA | O(u_1) O^\dagger(u_2) | AA \rangle \to \left( \frac{\pi}{l} \right)^{2x} e^{-\pi x (u_2 - u_1)/l}. \tag{1.8}$$

Here $|AA\rangle$ denotes the ground state on the strip with boundary condition $A$ on both sides. It simply corresponds to the absolute ground state, as remarked above. On the other hand, we may also calculate the Green function on the strip by inserting a complete set of states

$$\langle AA | O(u_1) O^\dagger(u_2) | AA \rangle = \sum_n |\langle AA | O | n \rangle|^2 e^{-E_n (u_2 - u_1)}. \tag{1.9}$$

Note that, in general, this sum must include all states with all possible boundary conditions on the bottom of the strip (but a fixed boundary condition $A$ on top of the strip). The lowest-energy intermediate state is the ground state with boundary conditions $A$ on the top and $B$ on the bottom. Thus,

$$x = \frac{l}{\pi} (E_{AB}^0 - E_{AA}^0). \tag{1.10}$$

(The other terms in the expansion of equation (1.9) correspond to the contribution of excited states with boundary conditions A and B.) Thus, we see that the scaling dimension of the operator which changes the boundary conditions from A to B is proportional to the ground-state energy with boundary conditions A, B, minus the ground-state energy with boundary conditions A, A.

So far, we have assumed that $\mathcal{O}^-$ produces the ground state in the sector of the Hilbert space with the modified boundary conditions A, B. It can also happen that the lowest-energy state produced by $\mathcal{O}$ is an excited state in this sector, $E_{AB}^1$. In this case

$$x = \frac{l}{\pi}(E_{AB}^1 - E_{AA}^0). \tag{1.11}$$

As we will see below, the deep core hole operator creates the ground state with a modified boundary condition, whereas the core hole conduction electron pair operator (which couples to the x-ray field) produces an excited state with a modified boundary condition.

In the next section we discuss the Fermi edge singularity for a Fermi liquid from this perspective, verifying equations (1.10) and (1.11). In section 3 we discuss it for a Luttinger liquid, i.e. an interacting one-dimensional electron system. In particular, we confirm the universal backscattering exponent recently obtained by Prokof'ev [11] by a different method. In section 4, we discuss the multi-channel Kondo–Anderson model from the perspective of boundary condition changing operators. In section 5, we discuss the connection between boundary condition changing operators and fusion, observing that the results of section 3 suggest a 'fusion rules' approach to the perfectly reflecting fixed point in a Luttinger liquid. We hope that this observation may lead to an exact solution of the mysterious finite-reflectance critical points discovered by Kane and Fisher [12].

## 2. The Fermi edge singularity in a Fermi liquid

In this section we briefly review the solution of the Fermi edge singularity in a Fermi liquid by Schotte and Schotte [13] and demonstrate that the relationship between scaling dimensions and energies of equations (1.10) and (1.11) is obeyed. For simplicity we assume a spherically symmetric dispersion relation and only s-wave scattering. The problem then reduces to a one-dimensional one defined on the half-line $r > 0$. It is convenient to work with left-moving fermions on the whole line by reflecting the right-movers to the negative axis

$$\psi_L(-r) \equiv \psi_R(r). \tag{2.1}$$

The left-movers are functions of $t + x$ only. (We set the Fermi velocity equal to one.) Hence, the Hamiltonian density becomes

$$\mathcal{H} = i\psi^\dagger \frac{d}{dx}\psi + \delta(x)V\psi^\dagger\psi bb^\dagger. \tag{2.2}$$

We have dropped the ubiquitous subscript L. We have also dropped the core hole energy $E_0$. It must be reinserted as a shift in the frequency upon Fourier transforming the expressions derived below. To make further progress we bosonize. We only need consider the left-moving half of a free massless boson field. Again we drop the L subscript. The left-moving fermion is represented in terms of a left-moving boson as

$$\psi \propto e^{i\sqrt{4\pi}\phi}. \tag{2.3}$$

The Hamiltonian becomes

$$\mathcal{H} = \left(\frac{\partial\phi}{\partial x}\right)^2 - \delta(x)\frac{V}{\sqrt{\pi}}\frac{\partial\phi}{\partial x}bb^\dagger. \tag{2.4}$$

The two Hamiltonians, $H_0$ and $H_1$ of equations (1.2) and (1.3), become

$$\mathcal{H}_0 = \left(\frac{\partial\phi}{\partial x}\right)^2 \tag{2.5}$$

and

$$\mathcal{H}_1 = \left(\frac{\partial\phi}{\partial x} - \frac{V}{2\sqrt{\pi}}\delta(x)\right)^2 \tag{2.6}$$

where we have dropped a cut-off-dependent ground-state energy contribution in $\mathcal{H}_1$ corresponding to a shift in the core hole energy $E_0$. This might seem dangerous since the difference in the ground-state energies of $H_0$ and $H_1$ will play a crucial role in what follows. However, it is only the *universal* part of this ground-state energy, determined by modular invariance, which will contribute. Note that $\mathcal{H}_1$ takes the same form as $\mathcal{H}_0$ when written in terms of a shifted field:

$$\tilde{\phi}(x) = \phi(x) - \frac{V}{4\sqrt{\pi}}\epsilon(x). \tag{2.7}$$

($\epsilon(x) = \pm 1$ for $x > 0$ and $x < 0$, respectively.) The fermion field is represented in terms of the boson as

$$\psi(x) \propto \exp(i\sqrt{4\pi}\phi(x)) = \exp(i\sqrt{4\pi}\tilde{\phi}(x) + iV\epsilon(x)/2). \tag{2.8}$$

Recalling from equation (21) that the field at $x < 0$ is the outgoing field, we see that

$$\psi_{\text{out}} = e^{2i\delta}\psi_{\text{in}} \tag{2.9}$$

with the phase shift

$$\delta = -V/2. \tag{2.10}$$

Thus, the bosonized model, with constant $V$, produces a $k$-independent phase shift equivalent to a boundary condition.

Schotte and Schotte [13] made the crucial observation that $\mathcal{H}_1$ is equivalent to $\mathcal{H}_0$ under a canonical transformation

$$\mathcal{H}_1 = U^\dagger\mathcal{H}_0U. \tag{2.11}$$

This follows from the commutation relations between the left-moving boson field $\phi$ and its derivative

$$\left[\frac{\partial\phi(y)}{\partial y}, \phi(x)\right] = \frac{-i}{2}\delta(x - y). \tag{2.12}$$

(The validity of this commutator can be checked by observing that, for a left-mover, $\partial\phi/\partial x = \partial\phi/\partial t$ and recalling that the full boson field is a sum of left and right parts, both of which contribute to the canonical commutation relations.) We find the operator $U$ satisfying equation (2.11) is

$$U = \exp(2i\delta\phi(0)/\sqrt{\pi}). \tag{2.13}$$

We note that $U$ can only be considered a unitary operator if we work in the extended Hilbert space which includes states with *all* possible boundary conditions. $U$ maps whole sectors of this Hilbert space, with particular boundary conditions, into each other.

Now that we understand the boundary condition changing operator $U$, it is straightforward to calculate Green's functions. To calculate the Green's function for the deep core operator $b(t)$ we use

$$b^\dagger(t) = e^{iHt}b^\dagger e^{-iHt} = e^{iH_0 t}b^\dagger e^{-iH_1 t}. \tag{2.14}$$

The second equality holds since the core hole must be present before $b^\dagger$ acts, but not after. The calculation then reduces to one in bosonic theory with the core hole operators eliminated:

$$\langle b^\dagger(t)b(0)\rangle = \langle e^{iH_0 t}e^{-iH_1 t}\rangle. \tag{2.15}$$

Now, using

$$e^{-iH_1 t} = U^\dagger e^{-iH_0 t}U \tag{2.16}$$

we obtain

$$\langle b^\dagger(t)b(0)\rangle = \langle U^\dagger(t)U(0)\rangle. \tag{2.17}$$

Using the free boson propagator

$$\langle\phi(t)\phi(0)\rangle = -\tfrac{1}{2}\ln(tD) \tag{2.18}$$

where $D$ is an ultraviolet cut-off, we obtain

$$\langle b^\dagger(t)b(0)\rangle \propto \frac{1}{t^{\delta^2/\pi^2}}.\overline{\phantom{--}} \tag{2.19}$$

Similarly, $b^\dagger(t)\psi(t,0)$ reduces to

$$U^\dagger(t)\psi(t,0) \propto \exp(i\sqrt{4\pi}(1-\delta/\pi)\phi(0,t)) \tag{2.20}$$

giving the x-ray edge exponent

$$\langle b^\dagger(t)\psi(t,0)\psi^\dagger(0,0)b(0)\rangle \propto \frac{1}{t^{(1-\delta/\pi)^2}}. \tag{2.21}$$

Fourier transforming, we obtain the singularity in the x-ray adsorption probability

$$\int dt\, e^{i(E+E_0)t}\langle b^\dagger(t)\psi(t,0)\psi^\dagger(0,0)b(0)\rangle \propto \frac{1}{(E+E_0)^\alpha} \tag{2.22}$$

with

$$\alpha = 1 - (1 - \delta/\pi)^2 = 2(\delta/\pi) - (\delta/\pi)^2. \tag{2.23}$$

(As mentioned above, strictly speaking, the core hole energy $E_0$ appearing on the right-hand side should be a renormalized one.)

   We now wish to verify equations (1.10) and (1.11), relating scaling dimensions to finite-size energy levels. Thus, we put the system in a box of length $l$ and impose the convenient boundary conditions

$$\psi_R(0) = \psi_L(0) \qquad \psi_R(l) = -\psi_L(l). \tag{2.24}$$

The former boundary condition is the result of the s-wave projection from three dimensions. The latter could arise, for example, from a vanishing boundary condition for the three-dimensional fermions on the surface of a sphere. It is actually more convenient to use the alternative formulation of the theory where we work with left-movers only on an interval of length $2l$ defined by equation (2.1). (When using this formulation we drop the subscript L, as above.) Then the first of equations (2.24) simply expresses the continuity of $\psi_L(x)$ at the origin, whereas the second becomes

$$\psi(-l) = -\psi(l). \tag{2.25}$$

In the free theory, from the bosonization formula of equation (2.3), taking into account the non-commutativity of the left-moving fields $\phi(x)$ and $\phi(y)$. This implies

$$\phi(-l) - \phi(l) = \sqrt{\pi} n \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{2.26}$$

The mode expansion for $\phi(x)$ with this boundary condition takes the form

$$\phi(x, t) = \sqrt{\pi} \frac{(t + x)}{2l} n + \sum_{m=1}^{\infty} \frac{1}{\sqrt{2\pi m}} \left[ \exp\left(-i\pi \frac{m}{l}(t + x)\right) a_m + \text{HC} \right]. \tag{2.27}$$

Here the $a_m$'s are boson annihilation operators. Substituting into the formula for the Hamiltonian with no core hole (equation (1.2)) we find the spectrum

$$E = \int_{-l}^{l} \left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{\pi}{l} \left[ -\frac{1}{24} + \frac{1}{2}n^2 + \sum_{m=1}^{\infty} m n_m \right] \tag{2.28}$$

where the $n_m$'s are the occupation numbers for the $m$th boson mode, $a_m^{\dagger} a_m$. The universal ground-state energy $E_0 = -\pi/24l$ is required by modular invariance [10]. It will play an important role in some of what follows. We can easily extend this to the theory with phase shift $\delta$. The first of equation (2.24) is now modified to

$$\psi_R(0) = e^{2i\delta} \psi_L(0). \tag{2.29}$$

We may again switch to a purely left-moving formulation by defining

$$\psi_L(-x) \equiv e^{2i\delta} \psi_R(x) \qquad (x > 0) \tag{2.30}$$

but now the boundary condition at $x = l$ becomes

$$\psi(-l) = -e^{2i\delta}\psi(l). \tag{2.31}$$

In bosonized form this becomes

$$\phi(l) - \phi(-l) = \sqrt{\pi}(n - \delta/\pi). \tag{2.32}$$

Note that this is the boundary condition obeyed by the shifted field $\tilde{\phi}$ of equation (2.7). $n$ is shifted to $n - \delta/\pi$ in the mode expansion of equation (2.27) and the spectrum becomes

$$E = \frac{\pi}{l}\left[-\frac{1}{24} + \frac{1}{2}\left(n - \frac{\delta}{\pi}\right)^2 + \sum_{m=1}^{\infty} mn_m\right]. \tag{2.33}$$

We can now read off the scaling dimensions from the spectra of equations (2.28) and (2.33) using equations (1.10) and (1.11). The operator $U$ or $b$ which creates a core hole takes the ground state into the phase-shifted ground state. Hence its scaling dimension is

$$x_b = \frac{l}{\pi}\left(E_\delta^0 - E_0^0\right) = \frac{\delta^2}{2\pi^2} \tag{2.34}$$

where the superscript denotes the ground state and the subscript denotes the phase shift. The operator $b\psi^\dagger$ maps the ground state into the state with phase shift $\delta$ and one conduction electron present, corresponding to $n = 1$ in equation (2.33). Hence,

$$x_{b\psi^\dagger} = \frac{l}{\pi}\left(E_\delta^1 - E_0^0\right) = \frac{1}{2}\left(1 - \frac{\delta}{\pi}\right)^2. \tag{2.35}$$

Since the two-point Green function for an operator of dimension $x$ scales as $t^{-2x}$, we see that equations (2.34) and (2.35) agree with equations (2.19) and (2.21).

Note that it was crucial to obtain the correct value for the relative ground-state energy with different boundary conditions in order to obtain the right scaling dimensions. These ground-state energies emerge naturally from the bosonized form of the theory, as derived above, but are not obvious from the fermion form. They can also be derived from a modular transformation, as discussed in [6].

## 3. Fermi edge singularity in a Luttinger liquid

In this section we derive the Fermi edge exponents for a Luttinger liquid, including backscattering. Our results, valid for arbitrary strength of the bulk interactions, agree completely with those of Prokov'ev [11] derived by a different method. Our approach is more closely related to that of [14]. We obtain the universal backscattering scaling dimension of $1/16$ from the dimension of the 'twist operator' in a free-boson theory which is related to the order parameter in the two-dimensional Ising model. In section 3.1 we discuss the case with zero backscattering. In section 3.2 we include backscattering. In section 3.3 we consider the case of electrons with spin and general forward and backward scattering.

The one-dimensional fermions consist of left- and right-movers, $\psi_L$ and $\psi_R$, with general bulk *Luttinger-liquid interactions. The (parity-invariant) core hole potential consists of* forward and backward scattering parts $V_f$ and $V_b$:

$$\mathcal{H}_{ch} = \delta(x)bb^\dagger[V_f(\psi_L^\dagger \psi_L + \psi_R^\dagger \psi_R) + V_b(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)]. \tag{3.1}$$

Once again, it is convenient to bosonize. Now we introduce both left- and right-moving bosons with

$$\psi_L \propto e^{i\sqrt{4\pi}\phi_L} \qquad \psi_R \propto e^{-i\sqrt{4\pi}\phi_R} \qquad \phi \equiv \phi_L + \phi_R. \tag{3.2}$$

The core hole interaction becomes

$$\mathcal{H}_{ch} = \delta(x)bb^\dagger\left[-\frac{V_f}{\sqrt{\pi}}\frac{\partial\phi}{\partial x} + \text{constant} \times V_b \cos(\sqrt{4\pi}\phi)\right]. \tag{3.3}$$

The Luttinger-liquid interactions just have the effect of renormalizing the Fermi velocity and rescaling the boson fields. Following the notation of [6], we introduce the 'compactification radius' $R$ for the boson field such that $R = 1/\sqrt{4\pi}$ in the non-interacting case. The interactions leave the bulk Hamiltonian in the non-interacting form after the rescaling

$$\phi \rightarrow \phi/\sqrt{4\pi}\,R. \tag{3.4}$$

Repulsive interactions lead to $R > 1/\sqrt{4\pi}$. The parameter $R$ is related to the parameter $g$ in [12] and the parameter $\phi$ in [11] (not to be confused with the quantum field in the present notation) by

$$g = e^{2\phi} = 1/4\pi R^2. \tag{3.5}$$

After this rescaling the full Hamiltonian becomes

$$\mathcal{H}_0 = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \delta(x)bb^\dagger\left[-\frac{V_f}{2\pi R}\frac{\partial\phi}{\partial x} + \text{constant} \times V_b \cos\frac{\phi}{R}\right]. \tag{3.6}$$

(We have again set the velocity equal to one.) We see that forward and backward scattering have very different effects. The forward-scattering term is always precisely marginal, leaving the theory non-interacting. It can be treated by the same methods as those used in the Fermi-liquid case discussed in the previous section. On the other hand, the backscattering term is relevant for $R < 1/\sqrt{4\pi}$ corresponding to repulsive interactions (and irrelevant in the other case). It introduces interactions into the boson model destroying its harmonic form.

It turns out to be convenient to use a somewhat different basis of fields to treat this problem. The same technique has been used previously in our treatment of this and other quantum impurity problems. The essential observation is that we may regard $\phi_R(-x) \equiv \phi_L'(x)$ as a second left-moving field. Such a transformation would presumably not be very useful if there were any bulk interactions in the bosonic formulation that mixed $\phi_L(x)$ and $\phi_R(x)$ since these would become non-local. But, importantly, the bulk part of the Hamiltonian is free and decouples into left and right terms. The only interactions occur at $x = 0$ and hence remain local. Thus we are free to make this transformation if we

wish. The theory is most conveniently studied in terms of parity-even and parity-odd linear combinations of these two left-moving fields:

$$\phi_{e,o} \equiv \frac{1}{\sqrt{2}}[\phi_L(x) \mp \phi_R(-x)]. \tag{3.7}$$

(Note from equation (3.2) that a parity transformation has the effect: $\phi_L(x) \leftrightarrow -\phi_R(-x)$, so the labels e and o are appropriate.) These fields obey the commutation relations

$$[\phi_e(x), \phi_e(y)] = [\phi_o(x), \phi_o(y)] = -\tfrac{1}{4}i\epsilon(x-y)$$

$$[\phi_e(x), \phi_o(y)] = -\tfrac{1}{4}i. \tag{3.8}$$

The Hamiltonian reduces to a sum of commuting terms involving the left-moving fields $\phi_{e,o}$ as

$$\mathcal{H} = \left(\frac{\partial \phi_e}{\partial x}\right)^2 + \left(\frac{\partial \phi_o}{\partial x}\right)^2 + \delta(x)bb^{\dagger}\left[-\frac{V_f}{\sqrt{2\pi}R}\frac{\partial \phi_e}{\partial x} + \text{constant} \times V_b \cos\frac{\sqrt{2}\phi_o}{R}\right]. \tag{3.9}$$

The usefulness of this peculiar change of variables is now evident; the Hamiltonian separates into two commuting terms for forward and backward scattering. The original fermion fields at the origin become

$$\psi_{L,R}(0) \propto \exp[i2\sqrt{2\pi}R\phi_e(0)]\exp\left[\pm i\frac{\phi_o(0)}{\sqrt{2}R}\right]. \tag{3.10}$$

The initial boundary conditions on the boson fields $\phi_e$ and $\phi_o$ simply specify that both fields are continuous at the origin. We will argue that the forward- and backward-scattering terms in the Hamiltonian are equivalent, at low energies, to modified boundary conditions on the fields $\phi_e$ and $\phi_o$, respectively. We will again be interested in calculating the dimensions of boundary condition changing operators. These will simply factorize into products of commuting operators acting on $\phi_e$ and $\phi_o$, respectively. All scaling dimensions will be sums of the forward- and backward-scattering parts from $\phi_e$ and $\phi_o$, respectively. Similarly, the finite-size spectrum will separate into a sum of even- and odd-excitation energies. For instance, the scaling dimension of the fermion field is

$$x_\psi = \pi R^2 + \frac{1}{16\pi R^2}. \tag{3.11}$$

(Note that this gives $x = \tfrac{1}{2}$ for $R = 1/\sqrt{4\pi}$, the free-fermion case.)

### 3.1. Forward scattering

Forward scattering can be treated by the same method reviewed in section 2. The boundary condition changing operator $U$ of equation (2.13) is

$$U = \exp(-iV_f\phi_e(0)/\sqrt{2\pi}R). \tag{3.12}$$

Thus, if we only have forward scattering, the deep hole annihilation operator has scaling dimension

$$x_b = \frac{V_f^2}{16\pi^3 R^2}. \tag{3.13}$$

Similarly the operator $b^\dagger \psi$ determining the x-ray exponent becomes

$$U^\dagger \psi_{\mathrm{L,R}} \propto \exp\left[i\left(\frac{V_f}{\sqrt{2}\pi R} + 2\sqrt{2}\pi R\right)\phi_e(0)\right]\exp\left[\pm i\frac{\phi_o(0)}{\sqrt{2}R}\right] \tag{3.14}$$

of dimension

$$x = \pi\left(R + \frac{V_f}{4\pi^2 R}\right)^2 + \frac{1}{16\pi R^2}. \tag{3.15}$$

Again this can be obtained from the finite-size spectrum with a phase-shifted boundary condition. We begin by defining the system on a circle of circumference $2l$; $-l < x < l$. Antiperiodic boundary conditions are imposed on the fermion fields: $\psi_{\mathrm{L,R}}(l) = -\psi_{\mathrm{L,R}}(-l)$. From bosonization formula (3.2), taking into account the commutation relations, it can be seen [6] that the corresponding boundary conditions on the fields $\phi_{e,o}$ are

$$\phi_e(l) - \phi_e(-l) = \pi\sqrt{2}Rn \qquad \phi_o(l) - \phi_o(-l) = \frac{m}{2\sqrt{2}R} \tag{3.16}$$

where $n$ and $m$ are arbitrary integers obeying

$$n + m = 0 \qquad (\mathrm{mod}\,2). \tag{3.17}$$

The forward-scattering interaction of equation (3.9) shifts the boundary condition on $\phi_e$ to

$$\phi_e(l) - \phi_e(-l) = \pi\sqrt{2}Rn + \frac{V_f}{2\sqrt{2}\pi R} \tag{3.18}$$

as can be seen by absorbing the potential scattering into a redefinition of the field $\phi_e$ as in equation (2.7). Equivalently, the modified boundary conditions on the fermion fields are

$$\psi_{\mathrm{L}}(0^+) = \exp[iV_f/4\pi R^2]\psi_{\mathrm{L}}(0^-)$$

$$\psi_{\mathrm{R}}(0^-) = \exp[iV_f/4\pi R^2]\psi_{\mathrm{R}}(0^+). \tag{3.19}$$

The mode expansions for the left-moving fields $\phi_{e,o}$ are

$$\phi_e(t+x) = \frac{(t+x)}{2l}\left(\sqrt{2}\pi Rn + \frac{V_f}{2\sqrt{2}\pi R}\right) + \cdots$$

$$\phi_o(t+x) = \frac{(t+x)}{2l}\frac{m}{2\sqrt{2}R} + \cdots \qquad n + m = 0 \qquad (\mathrm{mod}\,2) \tag{3.20}$$

where the $\cdots$ represent the harmonic modes. The finite-size spectrum is given by

$$E = \frac{\pi}{l}\left[-\frac{1}{12} + \pi\left(Rn + \frac{V_f}{4\pi^2 R}\right)^2 + \frac{m^2}{16\pi R^2} + \cdots\right] \qquad n + m = 0 \qquad (\mathrm{mod}\,2). \tag{3.21}$$

Note that the universal ground-state energy is doubled since there are two boson fields $\phi_e$ and $\phi_o$.

To make contact with the general formalism of section 1, we 'fold' the system about $x = 0$, regarding the left-moving fields at $x < 0$ as right-moving fields at $x > 0$. The above spectrum then corresponds to a trivial boundary condition at $x = l$ and a phase shift at $x = 0$ corresponding to the forward-scattering potential $V_f$. The ground-state energy ($n = m = 0$) $El/\pi \equiv x = V_f^2/16\pi R^2$ gives the dimension of the deep-hole creation operator $b$ in agreement with equation (3.13). The excited state with $n = m = 1$ gives the dimension of the operator $b^\dagger \psi$

$$x_{b^\dagger\psi} = \pi \left( R + \frac{V_f}{4\pi^2 R} \right)^2 + \frac{1}{16\pi R^2} \tag{3.22}$$

as in equation (3.15). Note that the energies are sums of even and odd parts and only the even part is modified by forward scattering.

## 3.2. Backscattering

Rather different methods are required to treat the backscattering interaction in equation (3.9). The Hamiltonian is presumably not unitarily equivalent to the free one. Furthermore, the backscattering interaction is relevant for $R < 1/\sqrt{4\pi}$ and is expected to renormalize to large values at low energies. Hence the scaling dimensions that we are after will not depend on the actual value of $V_b$. A way around these difficulties was found by Prokof'ev [11]. He argued that it should be possible to replace the cosine backscattering interaction by one quadratic in $\phi_o$ and exhibited a unitary operator, independent of $V_b$, which, at low energies, reduced the Hamiltonian to the non-interacting one. An alternative approach, which we use here, is to focus on the finite-size spectrum. As argued by Kane and Fisher [12] (see also [4]), $V_b$ renormalizes to $\infty$ corresponding to perfectly reflecting behaviour at low energies. The problem then is to find the scaling dimension of the operator which changes a perfectly transmitting boundary condition ($V_b = 0$) into a perfectly reflecting one ($V_b = \infty$). (For the moment, we set the forward scattering to zero.) As explained in section 1, this is equivalent to finding the ground-state energy with a perfectly transmitting boundary at one end of the system and a perfectly reflecting boundary at the other. This problem, which is essentially trivial, is solved in appendix B of [6] (see also [5]). It is simplest to go back to the original left- and right-moving bosons of equation (3.2) on the interval $-l \leqslant x \leqslant l$ with $-l$ and $l$ identified. The perfectly reflecting boundary condition at the origin is

$$\phi_L(0^\pm) = -\phi_R(0^\pm). \tag{3.23}$$

Note that the spatial component of the current is $J_1 \propto \partial\phi/\partial t$ and vanishes at $x \to 0$ as the origin is approached from either side. We may now use the trick of regarding the right-movers as reflected left-movers on *both* the positive and negative side of the origin. Hence, the system becomes equivalent to a single left-mover on an interval of length $4l$ with boundary conditions

$$\phi(4l) - \phi(0) = 2\pi nR. \tag{3.24}$$

The spectrum is

$$E = \frac{\pi}{2l} \left[ -\frac{1}{24} + 2\pi R^2 n^2 + \sum_{m=1}^{\infty} m n_m \right]. \tag{3.25}$$

Note the extra prefactor of $\frac{1}{2}$ due to the doubled length and the single ground-state energy term $-1/24$, since we effectively only have one boson field. From this formula and equation (3.21), we can read the scaling dimension of the operator which converts the transmitting boundary condition into a reflecting one:

$$x_b = -\frac{1}{48} + \frac{1}{12} = \frac{1}{16}. \tag{3.26}$$

Alternatively, in terms of the even and odd bosons, the perfectly reflecting boundary condition is

$$\phi_e(0^+) = \phi_e(0^-) \qquad \phi_o(0^+) = -\phi_o(0^-). \tag{3.27}$$

$\phi_e$ obeys a trivial boundary condition but $\phi_o$ obeys a twisted one. We see that the operator which changes the boundary condition from transmitting to reflecting acts trivially on $\phi_e$ but changes the sign of $\phi_o$. We see that inserting this boundary condition changing operator at the points $\tau_1$ and $\tau_2$ corresponds to inserting a cut along the $\tau$ axis between these two points. The field $\phi_o$ changes sign across the cut. This boundary condition changing operator is known as a *twist operator*. Its dimension, 1/16, is calculated in a somewhat different way in [15] where its relationship with the Ising-model order parameter is also discussed. We note that the term in equation (3.25) proportional to $n^2$ is identical to a term in the spectrum with no scattering, i.e. equation (3.21) with $V_f = 0$. This, together with the $m$ even terms in equation (25), can be identified as the contribution to the energy from $\phi_e$ which is unchanged by backscattering. On the other hand, the contribution of $\phi_o$ is changed significantly; in particular, it becomes independent of $R$.

The operator $b^\dagger \psi$ must correspond to a state with $n = 1$, since the $\phi_e$-dependence of this operator is simply that of the free fermion. Hence,

$$x_{b^\dagger \psi} = \frac{1}{16} + \pi R^2. \tag{3.28}$$

We may now include both forward and backscattering. A minor extension of the above calculation shows that the integer $n$, referring to the even sector, is shifted exactly as before, giving

$$E = \frac{\pi}{2l} \left[ -\frac{1}{24} + 2\pi R^2 \left( n + \frac{V_f}{4\pi^2 R^2} \right)^2 + \sum_{m=1}^\infty m n_m \right]. \tag{3.29}$$

Thus with both forward and backscattering, the dimensions of $d$ and $b^\dagger \psi$ become

$$x_b = \frac{1}{16} + \frac{V_f^2}{16\pi^3 R^2}$$

$$x_{b^\dagger \psi} = \frac{1}{16} + \pi \left( R + \frac{V_f}{4\pi^2 R} \right)^2. \tag{3.30}$$

These results agree completely with those of Prokof'ev (equation (24) and (30) of [11]).

### 3.3. Including spin

We now extend the model to include the electron spin. The core-hole part of the Hamiltonian is still given by equation (3.1) but with an implicit sum over spin indices. In general, we allow two Luttinger-liquid interactions which preserve only the $U(1)$ subgroup of the spin rotation group corresponding to rotations about the $z$-axis. We again bosonize and change variables from independent bosons, for spin up and down, to spin and charge bosons $\phi_s$ and $\phi_c$ (see [6] for further details). We again introduce parity-even and parity-odd left-moving fields for both spin and charge. The full Hamiltonian density, in bosonized form, becomes

$$
\mathcal{H} = \left(\frac{\partial \phi_{c,e}}{\partial x}\right)^2 + \left(\frac{\partial \phi_{c,o}}{\partial x}\right)^2 + \left(\frac{\partial \phi_{s,e}}{\partial x}\right)^2 + \left(\frac{\partial \phi_{s,o}}{\partial x}\right)^2
$$

$$
+ \delta(x) b b^\dagger \left[ -\frac{\sqrt{2} V_f}{\pi R_c} \frac{\partial \phi_{c,e}}{\partial x} + \text{constant} \times V_b \cos \frac{\sqrt{2}\phi_{c,o}}{R_c} \cos \frac{\sqrt{2}\phi_{s,o}}{R_s} \right]. \quad (3.31)
$$

Here the 'compactification radii' for charge and spin bosons are determined by the Luttinger-liquid interactions. They are related to the parameters of Kane and Fisher by

$$
g_{\rho,s} = 1/\pi R_{c,s}^2. \quad (3.32)
$$

Prokof'ev only considers the $SU(2)$ invariant case $R_s = 1/\sqrt{2\pi}$ and parametrizes the charge sector interactions by

$$
e^{2\phi} = 1/2\pi R_c^2. \quad (3.33)
$$

Note that the forward-scattering term only involves the even-charge boson $\phi_{c,e}$ and is, again, marginal. The backscattering term involves the odd-charge and odd-spin bosons. Under renormalization, we expect to generate pure spin and charge terms $\cos[2\sqrt{2}\phi_{0,s}/R_s]$ and $\cos[2\sqrt{2}\phi_{0,c}/R_c]$. The phase diagram is discussed extensively by Kane and Fisher [12] (see also [6]). Depending on the bulk interaction parameters, $R_c$ and $R_s$, there are four stable phases in which spin and charge are either perfectly reflected or perfectly transmitted. For some values of $R_c$ and $R_s$ more than one of these phases is stable for some range of scattering parameters and some unstable non-trivial fixed points occur at an intermediate scattering potential. We restrict our attention to the four stable phases. The fermion operators at the origin can be written

$$
\psi_{\uparrow,L} \propto \exp\left\{ i \left[ \sqrt{2\pi} R_c \phi_{c,e} + \sqrt{2\pi} R_s \phi_{s,e} + \frac{\phi_{c,o}}{\sqrt{2}R_c} + \frac{\phi_{s,o}}{\sqrt{2}R_s} \right] \right\}
$$

$$
\psi_{\uparrow,R} \propto \exp\left\{ i \left[ \sqrt{2\pi} R_c \phi_{c,e} + \sqrt{2\pi} R_s \phi_{s,e} - \frac{\phi_{c,o}}{\sqrt{2}R_c} - \frac{\phi_{s,o}}{\sqrt{2}R_s} \right] \right\}
$$

$$
\psi_{\downarrow,L} \propto \exp\left\{ i \left[ \sqrt{2\pi} R_c \phi_{c,e} - \sqrt{2\pi} R_s \phi_{s,e} + \frac{\phi_{c,o}}{\sqrt{2}R_c} - \frac{\phi_{s,o}}{\sqrt{2}R_s} \right] \right\}
$$

$$
\psi_{\downarrow,L} \propto \exp\left\{ i \left[ \sqrt{2\pi} R_c \phi_{c,e} - \sqrt{2\pi} R_s \phi_{s,e} - \frac{\phi_{c,o}}{\sqrt{2}R_c} + \frac{\phi_{s,o}}{\sqrt{2}R_s} \right] \right\}. \quad (3.34)
$$

We first consider the case of forward scattering only. The unitary operator which eliminates the forward-scattering term from the Hamiltonian is

$$
U = \exp(-i\sqrt{2} V_f \phi_{c,e}(0)/\pi R_c). \quad (3.35)
$$

Hence, with no backscattering

$$x_b = \frac{V_f^2}{4\pi^3 R_c^2}$$

$$x_{b\dagger\psi} = \frac{1}{4\pi}\left[\left(\frac{V_f}{\pi R_c} + \pi R_c\right)^2 + \left(\frac{1}{2R_c}\right)^2 + (\pi R_s)^2 + \left(\frac{1}{2R_s}\right)^2\right]. \qquad (3.36)$$

The backscattering term, depending on the values of $R_c$ and $R_s$, can twist the charge- or spin-odd boson; i.e. produce a perfectly reflecting boundary for charge and/or spin. From the discussion of the previous section, and from the finite-size spectra in [6], we see that each twist operator has dimension $1/16$. Hence, $x_b$ is increased by $1/16$ in the charge-transmitting, spin-reflecting or charge-reflecting, spin-transmitting cases and by $1/8$ in the charge- and spin-reflecting case. Only the last case was considered by Prokof'ev to correspond to $SU(2)$ symmetry $R_s = 1/\sqrt{2\pi}$ and a repulsive Luttinger-liquid interaction $R_c > 1/\sqrt{2\pi}$. Similarly, to the spinless case, and as seen from the finite-size spectrum in [6], the dimension of $x_{b\dagger\psi}$ is modified for relevant backscattering by the replacement of the term $1/16\pi R^2$ by $1/16$. This replacement is made for the $R_c$ term in the case where charge is perfectly reflected and is made for the $R_s$ term in the case where spin is perfectly reflected. In particular, when charge and spin are both perfectly reflected

$$x_{b\dagger\psi} = \frac{1}{4\pi}\left[\left(\frac{V_f}{\pi R_c} + \pi R_c\right)^2 + (\pi R_s)^2\right] + \frac{1}{8}. \qquad (3.37)$$

This is again in perfect agreement with Prokof'ev [11].

## 4. The Kondo effect

In the previous sections we have considered fairly trivial examples of boundary condition changing operators where the Hamiltonian $H_1$, which occurs after a deep hole is created, differs from the Hamiltonian $H_0$, which occurs without a deep hole by a simple potential scattering term. More general possibilities also exist in which the Hamiltonian, after the creation of a deep hole, differs from the unperturbed Hamiltonian by some non-trivial interaction terms involving additional dynamical degrees of freedom at the deep-hole location. An instructive example of this is provided by the Kondo–Anderson model, i.e. we consider the Kondo Hamiltonian but represent the impurity spin in terms of a deep-hole fermion operator with spin $b_\alpha$

$$S_{\text{imp}} = b^{\alpha\dagger}\frac{\sigma_\alpha^\beta}{2}b_\beta. \qquad (4.1)$$

The Hamiltonian is thus

$$H = \sum_k \epsilon_k a_{ik}^\dagger a_{ik} + E_0 b^{\alpha\dagger} b_\alpha + J b^{\alpha\dagger}\frac{\sigma_\alpha^\beta}{2}b_\beta \cdot \sum_{k,k'} a_{ik}^{\gamma\dagger}\frac{\sigma_\gamma^\delta}{2}a_{i\delta k'}. \qquad (4.2)$$

Here $i$ is a channel index which runs over $k$ values corresponding to the multi-channel Kondo effect. We may again consider the x-ray edge problem for this core hole state which

now carries spin. Again, the Hamiltonian commutes with the core hole occupation number. When the core state is either vacant or doubly occupied, the Hamiltonian reduces to the free one. When it is singly occupied the Hamiltonian reduces to the Kondo model. We have discussed the Kondo model extensively, emphasizing that, at low energies, the impurity spin is screened and all that remains is an effective conformally invariant boundary condition on the low-energy electronic degrees of freedom [9, 2]. Thus, at low energies, $b_\alpha$ is again a boundary condition changing operator. However, in the overscreened case ($k > 2s_{\text{imp}}$), this Kondo boundary condition is non-trivial leading to exotic effects like fractional ground-state degeneracy and non-trivial scaling laws. In all our previous discussion of the Kondo effect, we have only considered correlation functions involving the conduction-electron operators and the impurity-spin operator $S_{\text{imp}}$ and not the core hole operator $b_\alpha$. All the corresponding boundary operators live in the Hilbert space with a fixed Kondo boundary condition. Equivalently, they correspond to states in the finite-size spectrum with Kondo boundary conditions at both ends of the system. In this section we enlarge our discussion to include the boundary condition changing operator $b_\alpha$.

The boundary operator corresponding to $b_\alpha$ can be identified from the finite-size spectrum with Kondo boundary conditions at one end and free boundary conditions at the other. The ground state with free boundary conditions at both ends is the spin- and flavour-singlet charge-zero state. The Kondo boundary condition is obtained by fusion with the spin-1/2 flavour-singlet charge-zero operator and therefore the resulting spectrum contains the corresponding conformal tower [9]. Clearly the corresponding primary field $g_\alpha$ has the right quantum numbers to correspond to the deep-hole operator $b_\alpha$. Hence, we conclude that [9]

$$x_b = \frac{3/4}{2 + k} \tag{4.3}$$

the dimension of the Kac–Moody primary field $g_\alpha$. This result was suggested previously by Tsvelik in another context [16]. In the special case of one channel, $x_b = 1/4$. In this case, the Kondo fixed point is of Fermi-liquid type and simply corresponds to a $\pm\pi/2$ phase shift for spin-up or spin-down electrons. This corresponds exactly with the results in section 2. From equation (2.34), the scaling dimension is

$$x_b = \frac{1}{2}\left(\frac{\delta_\uparrow}{\pi}\right)^2 + \frac{1}{2}\left(\frac{\delta_\downarrow}{\pi}\right)^2 = \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{1}{4}. \tag{4.4}$$

We may also calculate the dimension of the 'x-ray operators' $b^{\alpha\dagger}\psi_{\beta i}$. These have charge $Q = 1$, transforms under the fundamental representation of the flavour group and have spin $j = 0$ or 1 depending on how the spin indices are contracted. A primary field of these quantum numbers is obtained by fusion of the $j = 1/2$ primary with the conformal tower of the free-electron operator with $j = 1/2$, $Q = 1$ and fundamental flavour representation for $k > 2$. We expect it to correspond to the 'x-ray operators' which, therefore, have dimension [9]

$$x_j = \frac{1}{4k} + \frac{k^2 - 1}{2k(2 + k)} + \frac{j(j + 1)}{2 + k} \tag{4.5}$$

with $j = 0$ or 1. In the special case $k = 1$, $x_0 = 1/4$. In this case, the $j = 1$ primary does not exist so $x_1$ must be a descendent with $x_1 = 5/4$. These results can be obtained from the phase-shift picture of equation (2.13), using

$$b_{\uparrow\downarrow} \propto \exp(2i(\delta_\uparrow\phi_\uparrow + \delta_\downarrow\phi_\downarrow)/\sqrt{\pi}) \propto \exp(\pm i\sqrt{2\pi}\phi_s)$$

$$\psi_{\uparrow\downarrow} \propto \exp(i\sqrt{2\pi}\phi_c)\exp(\pm i\sqrt{2\pi}\phi_s) \tag{4.6}$$

where $\phi_c$ and $\phi_s$ are charge and spin bosons and we have chosen $(\delta_\uparrow, \delta_\downarrow) = (\pi/2, -\pi/2)$ in $b_\uparrow$ and $(\delta_\uparrow, \delta_\downarrow) = (-\pi/2, \pi/2)$ in $b_\downarrow$.

## 5. Fusion

A useful method of generating new conformally invariant boundary conditions from known ones is by fusion [8]. We used this method to identify multi-channel Kondo boundary conditions [9]. On the other hand, we did not use this technique to find the boundary conditions corresponding to perfect reflection in a Luttinger liquid [6]. The insights gained from this paper suggest that such an approach is, in fact, possible. We hope that this will prove useful in solving a major open problem: the finite-reflectance critical points of Kane and Fisher [12].

If boundary condition $B$ is obtained from bondary condition $A$ by fusion with some operator $\mathcal{O}^a$ then the partition function on a finite cylinder with boundary condition $B$ at one end and an arbitrary boundary condition $C$ at the other, $Z_{BC}$, is determined by the same partition function with $B$ replaced by $A$, $Z_{AC}$, together with the fusion rule multiplicities $N_c^{ab}$. ($N_c^{ab}$ is a non-negative integer which specifies how many times the primary field $\mathcal{O}^c$ appears in the operator product expansion of $\mathcal{O}^a$ with $\mathcal{O}^b$.) The partition functions can be expanded in characters $\chi_a$ of the $a$th conformal tower,

$$Z_{AB} = \sum_a n_{AB}^a \chi_a \tag{5.1}$$

where the $n_{AB}^a$'s are non-negative integers. $Z_{BC}$ is determined by

$$n_{BC}^b = \sum_c N_c^{ab} n_{AC}^c. \tag{5.2}$$

As first shown by Cardy [8], under certain circumstances if an operator $\mathcal{O}^a$ changes boundary condition $A$ into boundary condition $B$ (i.e. produces the ground state on the strip with boundary condition $B$) then $B$ can be obtained from $A$ by fusion with the operator $\mathcal{O}^a$. We saw an example of this in the previous section where the $j = 1/2$ primary operator changed the free boundary condition into the 'Kondo boundary condition' and the latter was obtained from the former by fusion with this operator. Another example is provided by our discussion of forward scattering in section 3. With no scattering, general primary boundary operators are of the form

$$\exp\left[in2\sqrt{2\pi} R\phi_e(0)\right] \exp\left[im\frac{\phi_o(0)}{\sqrt{2}R}\right]. \tag{5.3}$$

After fusion with

$$U^\dagger = \exp(iV_f\phi_e(0)/\sqrt{2\pi}R) \tag{5.4}$$

we obtain the boundary operators

$$\exp\left[i(n2\sqrt{2\pi}R + V_f/\sqrt{2\pi}R)\phi_e(0)\right] \exp\left[im\frac{\phi_o(0)}{\sqrt{2}R}\right] \tag{5.5}$$

corresponding to the finite-size spectrum of equation (3.21).

We discovered in section 3 that the operator which changes a perfectly transmitting boundary condition into a perfectly reflecting one is the twist operator of dimension $x = 1/16$ for the odd boson $\phi_o$. Presumably, the latter boundary condition can be obtained from the perfectly transmitting one by fusion with this twist operator. (We have checked this explicitly for certain values of the compactification radius $R$.) Including spin, we have two different twist operators for $\phi_{c,o}$ and $\phi_{s,o}$. Presumably, fusion with some generalization of these twist operators will give the finite-reflectance critical point.

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